ON RANGE SPACE TECHNIQUES, CONVEX CONES, POLYHEDRA AND OPTIMIZATION IN INFINITE DIMENSIONS (with an ad hoc Functional Analysis introduction)

Paolo d'Alessandro

August 2022

Contents

0.1 INTRODUCTION

The main aim of this book is to present and expand on some recent author's research regarding minimum norm optimization for linear PDE, theory of polyhedra, optimization under polyhedral constraints and convex hypersurfaces, investigated from the range space point of view and in infinite dimensions .

Leitmotiv of this research is that all this topics are connected to the theory of convex cones and the fact that we never resort to differential techniques for optimization. All the results are founded on convexity, theory of convex cones and topology exclusively.

The author decided to include an introduction to functional analysis, not only for selfcontainedness, but also because a specific orientation to the topics dealt with in the research part of the book is needed.

Therefore the perspective is largely different from the many excellent textbooks on topology and functional analysis. In particular, we need to cover a good deal of non-standard material, which is essential for us. Such material include, for example, the theory of cone capping, the theory of support (including the Bishop Phelps Theorem), the range space approach to optimization, topics in lifting theory (lifting is indeed a range space technique) and much more.

Scattered in this introduction there is some original material. Among the various examples we cite the generalization of cone capping and of the Krein Milman Theorem for a the non compact case, which is of primary importance in this book, and the connection between extreme points, support, conical support and the closure of pointed cones.

Another peculiar feature of our introduction to functional analysis is that we aim to involve the larger possible audience.

We try to achieve this in various ways, illustrated below.

Narration has a relaxed pedestrian pace and many proofs and details, which are omitted in standard textbooks, are included. We try to never puzzle the reader skipping non obvious passages or with too many "it is readily seen" and the likes, which are too often not easily seen at all.

We also believe that it is more useful to stimulate further research, where one can find lot of matter for hard thinking, rather then ask readers time consuming and painful efforts to discover the well known.

Here and there, there are a few exercises, they are simple, aim at clarifying matters, and none of them is essential to read on.

Moreover, we strive to find and to give, whenever possible, the easiest proof of each theorem. This, of course, does not mean that we were able to dodge any difficult, long and involved proof, but we try hard to reduce them to the minimum possible.

In addition, we confine ourselves to core of a topic, whenever too many details seems not functional to our purposes. Nevertheless, not only we cover more than enough to gather the right preliminaries for the research part of the book, but, also, we believe we give the reader a solid base to undertake research in a wide range of fields.

Finally, we avoid to dwell on topics, which have been proved to be unnecessary to develop the required results.

A major example for this is Category Theory.

Although V. Ptak proved in 1958 that a group of fundamental theorems of vector topology (the Closed Graph Theorem, the Inverse Map Theorem and the Open Mapping Theorem, usually called Grand Theorems) do not depend on Category, some textbook keep using category theory, whereas it would be more instructive, we believe, to dwell on Ptak's theory. Here we will give a succinct account (The reader can find an excellent account in [13]). However, we will derive all the Grand Theorems in Banach spaces, without any use of Category theory.

This attitude to make readers' life easier is rare but not unique, and this is of comfort to the author. There are some examples of this in the references, like the excellent books of P.R. Halmos, with their reader friendly clarity, and [79], with its welcome conversational emphasis on intuition and motivations. Besides, Halmos himself in [80] argued (rightfully in our opinion) that mathematics is a form of art, and although any art has his own more or less difficult language, we believe we should allow the largest possible public to enjoy its beauty with the less possible effort.

In what follows it is assumed familiarity with rudiments of measure theory (see e.g. [20]). An Appendix for a future edition covering these topics is in the making, but is not included for timeliness reasons. Also some facts regarding real or complex valued function spaces (e.g. Liouville Theorem) are recalled without a proof.

Regarding references, nowadays in the internet era, the author confesses that sometimes it has been at loss giving proper credits, which would have been not only dutiful, but also a great pleasure. For example, there are many lecture notes downloadable in Chapters in the net, and it is impossible most of the times, even looking at metadata, to trace back the authors. A similar situations prevails for the very useful sites on the net of Mathematical questions and answers.

***

We give a basic overview of General and Vector Topology, Banach and Hilbert spaces.

The Hahn Banach Theorem is essentially a Theorem based on the theory of convex cones and on radiality, and is independent of vector topology (it becomes a topological Theorem by the connection between radial kernels and interiors).

Given that convex cones and radiality are central ingredient of our development, we follow the excellent classic [10] in giving the non-topological version of the Hahn Banach Theorem, not only because it provides a better insight, but, also, because it allows us to cover material that is too essential for us to be overlooked.

Extreme points are of utter importance throughout the book. In this respect a fundamental ingredient, for example in norm optimal control, is the round of ideas that connects extreme points to tangent cones, to support and to conical support and, finally, to a crucial theorem on the closure of pointed cones, which was introduced in [1] (and is valid in linear topological spaces). This connection produce a sufficient condition of support for void interior closed convex sets, which is instrumental in our solution of many optimization problems. For their importance these topics are given first within locally convex spaces and then revisited in more detail in Hilbert spaces.

In a subsequent Chapter we will give, in the Hilbert space environment, a necessary and sufficient condition of support for void interior closed convex sets.

Interestingly, some problems in the theory of polyhedra and optimization over polyhedra are solved considering not only extreme points, but also a different kind of conical support points, which we call *beacon points*.

We give an overview of strongly continuous semigroup, which serves mainly to introduce the mild solution of abstract linear differential equation. This is because the dynamic optimization problem we will deal with (norm optimal problems), assume as starting point the explicit solution (namely the mild solution) of the differential equation.

This is a crucial aspect, as also underlined in [25], for the optimal control problems like norm optimality under linear PDE, because the investigation of these problems not only requires that an explicit solution be known, but, also, it would be impossible without exploiting properties of the solution, like, for example, continuity in time, which is required to specify target vectors at a given time. Of course, this would be impossible if the solution were defined a.e..

Naturally, for specific examples, one has to fit in the abstract differential model, specific cases of PDE. In [25] one can find many examples of specific partial differential equations. There are cases where this require exploiting the machinery of Sobolev spaces.

This is not of interest here, since we are focused on proving the Maximum principle and specify the extent to which it is applicable, by means of the mentioned machinery about extreme points, closure of pointed cones and support. In this book we expand on [1] and [32].

We introduce infinite dimensional polyhedra already at the level of locally convex spaces, while a full fledged theory of polyhedra and optimization over polyhedra is presented in the separable Hilbert space environment.

Since there is a strict connection between range space point of view, the theory of polyhedra and cone capping, we include, to start with, the classical theory of cone capping (which is based on compactness) already at the locally convex level and, subsequently, we go more in depth in the Hilbert space environment.

However, the classical theory of cone capping is not adequate to deal with infinite dimensional polyhedra because in this case weak compactness cannot be achieved, and, therefore, we introduce a generalization of this theory, allowing for a special class of unbounded (non weakly compact) capped cones. This generalization is presented for the case of Hilbert spaces. It introduces the concept of *generalized polytopes* (g-politopes: different from what is found in the literature, as will be explained) and includes a *generalization of the Krein Milman theorem*.

We will revert to the classical cone capping theory in the part devoted to convex surfaces, because in this case classical capping is adequate.

Another highlight in the theory of convex cones, regards convex cone decomposition (which is another essential ingredient for developing both the theory of polyhedra and of convex surfaces). A non pointed convex is a cone which contains a non-trivial linear subspace. The maximal linear subspace contained in the cone is called *the lineality space of the cone.* It is well known, in finite dimension, that any non-pointed cone can be expressed as the sum of the lineality space plus a certain determined pointed cone. This decomposition was extended in infinite dimensions previously by the author in the case of closed lineality subspace. Here not only we improve on this decomposition, but also propose a different technique for dealing with the decomposition problem for non closed cones with non closed lineality space. Such decomposition will allow us to prove *a necessary and sufficient condition of support for void interior closed convex sets.*

***

Before starting we feel we should give a little motivation for the range space point of view, which informs extensively this book. We will start, with an apology, with very trivial examples, but they are useful anyway, to introduce and explain the round of ideas, which is then applied to non-trivial problems.

One basic aspect of range space is the concept of lifting. Consider R^n. It is a function space with domain $\{1, .., n\}$ and range R. It is difficult to think of a domain farther away from the idea of linear space. And yet the property of being a linear space is *lifted* from the range space (R) to the function space R^n.

Speaking of non trivial liftings, an example of results, that we will include, because they are instrumental within the research part of the book, is the lifting of uniform convexity of a Banach space (acting as range space) to uniform convexity of spaces of Bochner integrable functions over such Banach space.

Naturally, one cannot resist the temptation to say that the passage from a Riemann integral to a Lebesgue integral is exactly a passage from a domain space point of view to a range space point of view. Instead of partitioning the *domain* to define integral sums, the idea is to weight *values* with the measure of the sets where they are attained.

Regarding optimization, we start, with an apology, from another very trivial example. A diligent student of calculus is asked to find the maximum of the function sin in $[0, 2\pi]$. So he/she computes the first and second derivative of the function (well suppose the function had no derivative, then what) and everybody knows how he/she will conclude that the maximum is at $\pi/2$ and the maximum value is 1.

Let's revisit this example from the range space point of view. The range of the sin function is $[-1, 1]$ and therefore the maximum value is 1. End of exercise.

Here we meet a fundamental feature of the range space approach. We don't (at least initially) determine a solution, we determine the maximum value (if it exists at all). Of course if one wants a solution all we have to do is to compute an inverse image or to invert the function according to the cases.

Now, passing from a trivial example to research, one of the topics we will deal with is Linear Programming in infinite dimensions, naturally dealt with from the range space point of view.

Part of the present book is a continuation of the book [5], dealing with the finite dimensional polyhedra studied from the range space point of view. Once we switch to the range space approach we immediately discover that the natural issues (like feasibility, structural theory and optimization) are a whole matter of theory of convex cones and cone capping.

Incidentally, we will give a review of finite dimensional theory here for various reasons. Firstly, for completeness. Secondly for comparison with the infinite dimensional case. And thirdly because, as we will show, the two cases are connected by a technique of finite dimensional approximations.

One of the main goal of the research part is to show how the finite dimensional theory generalizes to separable Hilbert spaces. This generalization considers countable intersections of closed semispaces (instead of finite intersections) with the consequence that arbitrary closed convex sets become polyhedra.

***

Our final remark are about two papers by the author, regarding infinite dimensional polyhedra, which are completely superseded by the present book. The author discovered to have a cancer in 2010, and, for this reason, after a two year sabbatical, retired in 2012. Since then his life has been repeatedly in jeopardy. The sense of uncertainty about survival time has lead to two premature submissions, regarding [8] and [4]. This two papers, especially the first contains some errors, but the same material is treated, corrected, improved and expanded in this book. Thus it is recommended that, for these topics , exclusive reference is made to the present book, which overrides both of these papers.

0.2 ABBREVIATIONS and NOMENCLATURE

rhs=right hand side
lhs=left hand side
s.t.=such that
iff=if and only if
N&S=necessary and sufficient
wrg=without restriction of generality
lub=least upper bound
glb=greatest lower bound
chain, tower and nest are the same and indicate a linearly ordered set
ls=linear space
lts= linear topological space
lcs=locally convex linear topological space
lf=linear functional
clf=continuous linear functional
nb=neighborhood
DCT=Dominated Convergence Theorem
MCT= Monotone Convergence Theorem
PDE= Partial Differential Equation

Semispace and halfspace are used interchangeably to denote sets of the form $\{x : f(x) \leq a\}$, or else $\{x : f(x) \leq a\}$, where f is a real valued linear functional.

The specifications *semi*, *pseudo* and *quasi* are used interchangeably. For example pseudonorm and seminorm are the same thing. By the same token, pseudoboundary and quasiboundary are the same thing.

There are some cases where we adopt, so to speak, popular poetical licences, which in practice simplify exposition. For example, one cannot say that the set A contains a point x, because what is meant correctly is $A \supset \{x\}$, which is equivalent to $x \in A$. However, if it is understood what is meant, it is often simpler to say A contains x rather that x belong to A. Thus we too follow this non rigorous habit. Similarly, while a sequence $\{x_i\}$ is a function, we indicate with the same symbol the range of the sequence, leaving to specifications and context to distinguish which is which.

0.3 LIST OF SYMBOLS

$\exists!$ means exists and is unique.

If A and B are a subsets of a space E, then $A \backslash B$ is their difference set